Apportionments with minimum Gini index of disproportionality

Daniele Pretolani

DISMI, Università di Modena e Reggio Emilia
e-mail: daniele.pretolani@unimore.it

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Goal

Distribute parliamentary seats among *parties*, proportionally to the *votes* cast for each party

. . . but actual electoral systems *almost never* do that!

- *district based plurality or mixed systems (France, GB)*
- *multi-level systems (Switzerland, Germany)*

Distribute seats among the *states* of a federal system, proportionally to the *population* of each state

- EU Parliament, US House of Representatives
- the latter is the "pure" version of the problem [Balinski and Young (1981)]

Some history

Very first use of the *Presidential Veto* in US history

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Some history

Very first use of the *Presidential Veto* in US history

- *George Washington* after 1790 census
- turned back to the method by *Thomas Jefferson*
- discarding the method by *Alexander Hamilton*

Several methods adopted since then, with several court cases, and a still ongoing debate

The Apportionment Problem

Requires to map *large* numbers (votes, population) into *small* numbers (seats)

- *distortion is unavoidable*

A sort of *vector scaling* problem: scale a set of integers so that their sum is a given value

Prototype example of *fair allocation of a discrete finite resource*

The Apportionment Problem

Requires to map *large* numbers (votes, population) into *small* numbers (seats)

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A sort of *vector scaling* problem: scale a set of integers so that their sum is a given value

Prototype example of *fair allocation of a discrete finite resource*

What is fair?

What is Fair?

- no unique answer, several (often conflicting) answers
- several methods developed in the last couple of centuries
- two main classes of methods (or combinations of them)
  - *quotient methods*
  - *divisor methods*
- the stress is on *choosing a method*, not on choosing an apportionment

Two streams of research

- the axiomatic approach
- the optimization approach

What is Fair?

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Two streams of research

- *the axiomatic approach*
- the optimization approach

The Optimization Approach

Characterize a method based on the *inequality index* it minimizes

- an inequality (or *disproportionality*) index is a (mathematically sound) measure of the "unfairness" introduced by an apportionment
- several (classes of) inequality indexes have been proposed
- for most indexes we know the corresponding "minimizing" method
  - *move from choosing a method to choosing an index*

A strong characterization [Simeone *et al.* 1999]
A proportional method is a *procedure* solving an underlying *integer optimization* (IP) problem, where the objective function (the "hidden criterion") is an inequality index

Most classic methods are greedy procedures minimizing a convex separable inequality index

The Optimization Approach (2)

Beyond characterization, *new* proportional methods can be *defined* following the optimization approach

- choose an inequality index as objective function
- possibly insert constraints enforcing specific properties *(quota property, "degressive proportionality",...)*

Among others, the Gini Index is suggested as a possible choice

> *"Electoral formulas minimizing entropy, the Gini concentration index, standard deviation, and other common indexes of inequality could be just as good as other consolidated methods"*
>
> [Simeone *et al.* (1999), Ch. 6.4]

We pursue this suggestion in this work

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The Gini Index

A measure of statistical dispersion due to Corrado Gini ($\approx$ 1912)

An index of *concentration*, i.e., unequal distribution of income or wealth among a population

- commonly adopted in welfare economics, social sciences, and *many* other disciplines
- often used to compare disproportionality of apportionments
- ranges between zero (perfect egalitarianism) and one (perfect concentration)

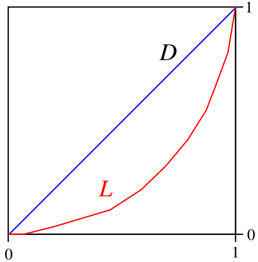
Two standard definitions

- *algebraic:* normalized average of the wealth differences *(to be used later)*
- *geometric:* given in terms of *Lorenz curves*

Lorenz Curves

One of the first historical attempts ($\approx$ 1905) to represent and compare wealth distributions

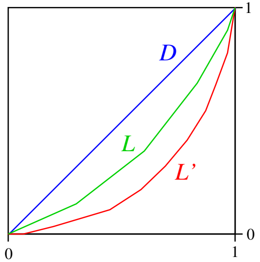
- plot of *cumulative wealth*, normalized within a unit square
- individuals sorted *left to right in non-decreasing order of wealth*



The Lorenz curve L is convex and lies below diagonal D

Lorenz Curves (2)

The more L is far from D, the more the distribution is unequal (concentrated)

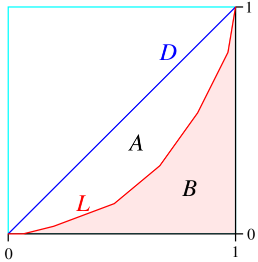


- L' is "more unequal" than L
- for a *perfectly egalitarian* distribution the Lorenz curve is D

The Gini Index: Geometric Definition

Consider a wealth distribution with corresponding Lorenz curve L

- A is the area between D and L
- B is the area below L



The corresponding Gini index is

$$G = \frac{A}{A+B} = 2A = 1 - 2B.$$

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Overview

- restrict analysis to apportionments *satisfying quota*
  - *an apportionment violating quota may give a lower G*
- a new *quotient method*, exploiting the optimization approach
- the underlying IP problem has a *binary knapsack* structure
- geometric approach: maximize area B
- express B as a *quadratic function* of the knapsack variables
- use an efficient solver (or tight ILP formulations) for the resulting (supermodular, binary) quadratic knapsack instance

Annals of Operations Research 215 No. 1 (2014)
Special Issue in memory of Bruno Simeone

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Scenario

- A set of n *states* $\{1, 2, \ldots, n\}$
- The *population vector* $v = [v_1, v_2, \ldots, v_n]$
- The total population $V = \sum_{i=1}^{n} v_i$
- The number of seats S

▶ The output apportionment $s = [s_1, s_2, \ldots, s_n]$ $(S = \sum_{i=1}^{n} s_i)$

Examples

- *EU Parliament* (2011): $n = 27$, $S = 751$, $V = 501,103,425$
  (. . . before brexit. . .)
- *US House of Representatives* (2010): $n = 50$, $S = 435$, $V = 309,183,463$

A generic quotient method

Define the *natural fractional quota* q_i of state i:

$$q_i \quad = \quad \frac{S \cdot v_i}{V} \quad = \quad \lfloor q_i \rfloor + r_i \quad = \quad u_i + r_i$$

where r_i is the *remainder* of state i $(0 \leq r_i < 1)$

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Quotient Method

(i) assign to each state i the "minimum quota" u_i

(ii) assign one seat more to K different states

$$K = S - \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} r_i$$

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Hamilton's Method ("largest remainders"): assign the additional seats to the states with the K largest remainders

Quotient methods: remarks

- the case of null remainders can be ignored: $0 < r_i < 1$
- we have $0 < K < n$ and we can expect $K \approx n/2$
  - EU (2011): $n = 27, \ K = 14$
  - US (2010): $n = 50, \ K = 23$
  - US (2000,1990): $n = 50, \ K = 26$

Quotient methods: remarks

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  - US (2010): $n = 50, \;\; K = 23$
  - US (2000,1990): $n = 50, \;\; K = 26$

Quota Property: apportionment s *satisfies quota* *iff.* for each i

$$q_i - 1 \;<\; s_i \;<\; q_i + 1$$

Theorem [Balinski and Young (1982)]
quotient methods are the only ones guaranteed to satisfy quota

Quotient method: knapsack structure

Let us introduce a vector of binary variables $x \in \{0, 1\}^n$

In a quotient method $s = u + x$ with the knapsack-like constraint

$$\sum_{i=1}^{n} x_i \quad = \quad K$$

(a pretty easy case of knapsack, solvable in $O(n)$ for a linear O.F.)

We shall need a **quadratic** objective function

$$x^{\mathrm{T}} Q x \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j$$

with Q an $n \times n$ matrix (with *nonnegative* entries)

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Apportionment and Welfare

Apportionments as wealth distributions

- S is the total wealth shared by a population of V individuals
- Each state i owns a total wealth s_i
- Each citizen of state i owns the same wealth $w_i = s_i / v_i$

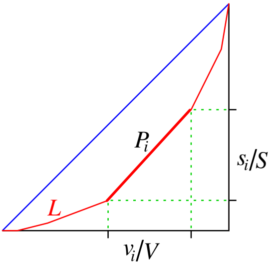
Remarks

- the “wealth” w_i is often referred to as the *voting power*
- the reciprocal $1/w_i = v_i/s_i$ is often referred to as the *district size* *(“how many citizens it takes to gain one seat”)*

Apportionment and Welfare (2)

In the Lorenz curve a state i is represented by a *single linear piece* P_i

- horizontally spanning v_i/V (the population share)
- vertically spanning s_i/S (the share of seats)



Pieces appear in L, left to right, in increasing order of voting power w

Apportionment and Welfare (3)

The Gini index G is used as a measure of unequal distribution of voting power

In our quotient method L, B and G depend on x

We choose x to maximize B, that is, to minimize G

Goal: express the area B as a function of x

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Unfortunately, x acts on L (and B) in a rather subtle and criptic way

Apportionment and Welfare (4)

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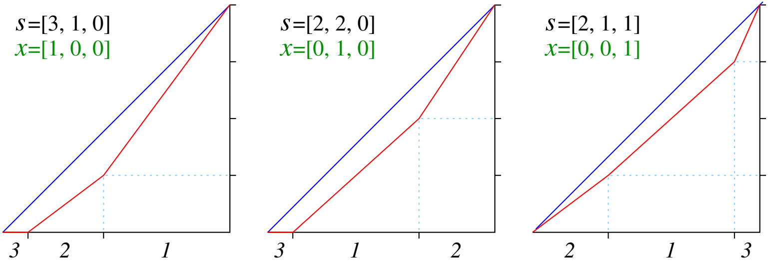
Example: $V = 9$, $S = 4$, $v = [5, 3, 1]$; $u = [2, 1, 0]$, $K = 1$



(states listed on the horizontal axis in increasing w order)

Apportionment and Welfare (4)

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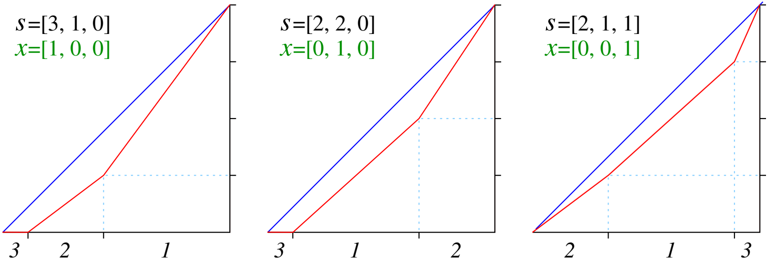

(states listed on the horizontal axis in increasing w order)

Try a successive decomposition approach

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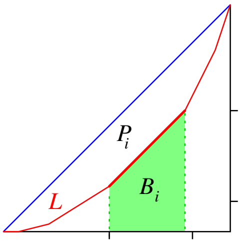
Decomposition: B as a function of x

Step 1: cut B into n vertical “slices”, one for each state i

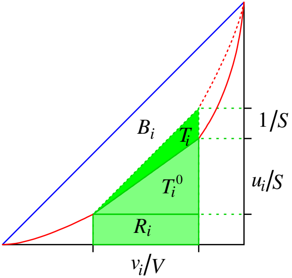


Slice B_i lies under linear piece P_i

$$B = \sum_{i=1}^{n} B_i$$

Decomposition: B as a function of x (2)

Step 2: decompose each slice B_i



$T_i^0 = \frac{v_i \cdot u_i}{2SV}$: *constant, independent of x*

$T_i = \frac{v_i \cdot x_i}{2SV}$: *linear in x*

$R_i =$ **?** *depends on "poorer" states!*

$$B = \sum_{i=1}^{n} B_i = \sum_{i=1}^{n} \left(T_i^0 + T_i \cdot x_i \right) + \sum_{i=1}^{n} R_i$$

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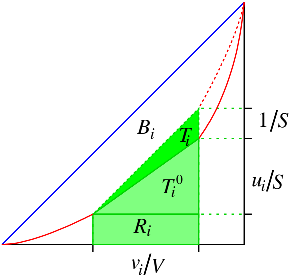

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Next step: decompose the **sum** of the R_i (not each R_i)

Decomposition: terminology

Notation

- assume *w.l.o.g.* $i < j \iff \frac{u_i}{v_i} < \frac{u_j}{v_j}$
- given $s = u + x$, if i appears in L on the left of j denote

$$i \lhd^x j \iff \frac{s_i}{v_i} < \frac{s_j}{v_j} \iff w_i < w_j$$

"i precedes j (given x)"

- the following holds for each each i

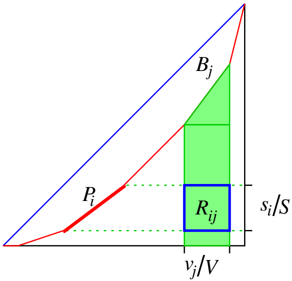
$$\frac{u_i}{v_i} < \frac{q_i}{v_i} = \frac{S}{V} < \frac{u_i + 1}{v_i}$$

Decomposition: R_i as a function of x

Given two states, $i < j$, two *mutually exclusive* cases can occurr

- $i \lhd^x j$: i induces a slice of R_j with area $R_{ij} = \frac{v_j \cdot s_i}{SV}$
- $j \lhd^x i$: j induces a slice of R_i with area $R_{ij} = \frac{v_i \cdot s_j}{SV}$

The picture shows the case $i \lhd^x j$



$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij}$$

Decomposition: R_{ij} as a function of x

Given two states $i < j$ we have by assumption $\frac{u_i}{v_i} < \frac{u_j}{v_j} < \frac{S}{V}$

We need the relation between $\frac{u_i + 1}{v_i}$ and $\frac{u_j + 1}{v_j}$

Again, two *mutually exclusive* cases can occurr:

Case 1 $\frac{S}{V} < \frac{u_i + 1}{v_i} < \frac{u_j + 1}{v_j}$

Case 2 $\frac{S}{V} < \frac{u_j + 1}{v_j} < \frac{u_i + 1}{v_i}$ *(implies $v_i < v_j$)*

For each pair $i < j$ we know which case arises

For each value of x_i and x_j we deduce $\lhd^x$ and compute R_{ij}

R_{ij} as a function of x: Case 1

x_i	x_j	w_i	w_j	$\lhd^x$	R_{ij}
0	0	u_i/v_i	u_j/v_j	$i \lhd^x j$	$u_i \cdot v_j/SV$
0	1	u_i/v_i	$(u_j+1)/v_j$	$i \lhd^x j$	$u_i \cdot v_j/SV$
1	0	$(u_i+1)/v_i$	u_j/v_j	$j \lhd^x i$	$u_j \cdot v_i/SV$
1	1	$(u_i+1)/v_i$	$(u_j+1)/v_j$	$i \lhd^x j$	$(u_i+1) \cdot v_j/SV$

By setting $a_{ij} = u_j \cdot v_i - u_i \cdot v_j$ we obtain

$$R_{ij} = u_i \cdot v_j/SV + (a_{ij}/SV)x_i + \Big((v_j - a_{ij})/SV\Big)x_i x_j$$

it is easy to check that $a_{ij} \geq 0$ *and* $(v_j - a_{ij}) \geq 0$

R_{ij} as a function of x: Case 2

x_i	x_j	w_i	w_j	$\lhd^x$	R_{ij}
0	0	u_i/v_i	u_j/v_j	$i \lhd^x j$	$u_i \cdot v_j/SV$
0	1	u_i/v_i	$(u_j+1)/v_j$	$i \lhd^x j$	$u_i \cdot v_j/SV$
1	0	$(u_i+1)/v_i$	u_j/v_j	$j \lhd^x i$	$u_j \cdot v_i/SV$
1	1	$(u_i+1)/v_i$	$(u_j+1)/v_j$	$j \lhd^x i$	$(u_j+1) \cdot v_i/SV$

Again, with $a_{ij} = u_j \cdot v_i - u_i \cdot v_j$ we obtain

$$R_{ij} = u_i \cdot v_j/SV + (a_{ij}/SV)x_i + (v_i/SV)x_i x_j$$

also in this case $a_{ij} \geq 0$

Putting everything together…

$$
\begin{aligned}
B &= \sum_{i=1}^{n} B_i \\
&= \sum_{i=1}^{n} \left(T_i^0 + T_i \cdot x_i \right) + \sum_{i=1}^{n} R_i \\
&= \sum_{i=1}^{n} \left(T_i^0 + T_i \cdot x_i \right) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij} \\
&= \sum_{i=1}^{n} \left(T_i^0 + T_i \cdot x_i + \frac{1}{SV} \sum_{j=i+1}^{n} u_i \cdot v_j + a_{ij} \cdot x_i + \delta_{ij} \cdot x_i \cdot x_j \right) \\
&= C + x^{\mathrm{T}} Q x
\end{aligned}
$$

$\delta_{ij} = (v_j - a_{ij})$ for Case 1, $\delta_{ij} = v_i$ for Case 2

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The Underlying IP Problem

Quadratic Knapsack Problem (QKP)

$$(P) = \begin{cases} \max \; x^{\mathrm{T}} Q x \\ \displaystyle\sum_{i=1}^{n} x_i \quad = \quad K \\ x \in \{0,1\}^n \end{cases}$$

Q has non-negative entries, which complies with the original definition of (QKP) ("supermodular" quadratic knapsack) [Simeone (1979), Gallo, Hammer and Simeone (1980)]

Quite particular case, with a "cardinality" constraint
"p-dispersion" problem [Pisinger (2006)]

Quadratic Knapsack

The general case is much harder than its linear counterpart

Instances derived from real apportionment problems are *small*
the largest we tried (US 2000) has $n = 50$ *and* $K = 26$

We tried two solution methods, based on the same approach
[Caprara, Pisinger and Toth (1999)]

- C language implementation of procedure *quadknap*
  available from `http://www.diku.dk/~pisinger/`
- Enhanced ILP formulation (via *Xpress-Mosel*)
  (≈ 0.5 *seconds: about ten times slower than "quadknap")*

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Note: for each instance we solved $K + 1$ QKPs, applying a trivial branching method to test *uniqueness* of the optimum; CPU times are those of "quadknap"

Results

Results for US House of Representatives ($n = 50,\ \ S = 435$)

Year	*Tot. Pop.*	*K*	min G	current G	*cpu sec.*
1990	249, 022, 783	26	0.021594	0.021812	1.33
2000	281, 424, 177	26	0.020298	0.020308	1.30
2010	309, 183, 463	23	0.020862	0.020862	1.29

Very small differences: $< 1\%$*!*

- 2000: one seat from the larger to a small state
- 1990: three seats from smaller to larger states

Results

Results for the EU Parliament ($n = 27, \; S = 751$)

Year	*Tot. Pop.*	K	min G	current G	*cpu sec.*
2011	$501,103,425$	14	0.0055851	0.128271	0.14

The *big* difference (current solution about 20 times larger!) is due to the “degressive proportionality” rules in current EU treaties

Index is about four times smaller than US House *(half the states, twice the seats)*

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The General Case

For an apportionment s, we can write G as follows:

$$G = \frac{1}{SV} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |v_j \cdot s_i - v_i \cdot s_j|$$

- can be obtained rather easily from the *algebraic* definition
- can be obtained generalizing the geometric approach, but some more algebra is involved

We have standard LP techniques to deal with the absolute value

General Case: the IP Problem

$$(IP) = \begin{cases} \min \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij}^{+} + d_{ij}^{-} \\ \sum_{i=1}^{n} s_i = S \\ v_j \cdot s_i - v_i \cdot s_j = d_{ij}^{+} - d_{ij}^{-} \qquad 1 \leq i < j \leq n \\ d^{+}, d^{-}, s \text{ integer} \end{cases}$$

However, IP is not directly solvable in reasonable time
(rapidly running out of memory for the US instances)

Possible directions

- find *tight* lower and upper bounds for each s_i
- try a binary formulation: *Multiple Choice Knapsack*

A possible approach: Multiple Choice Knapsack

Rewrite each variable s_i as a weighted sum of binary variables

$$s_i = \sum_{j \in J(i)} j \cdot x_{ij}$$

Then the knapsack constraint breaks into $n + 1$ constraints

$$\sum_{i=1}^{n} \sum_{j \in J(i)} j \cdot x_{ij} = S$$

$$\sum_{j \in J(i)} x_{ij} = 1 \qquad 1 \leq i \leq n$$

Needs a *smart* linearization for the absolute value $|v_j \cdot s_i - v_i \cdot s_j|$

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Degressive Proportionality

The Treaty of Lisbon sets a “degressive proportionality” condition (DP) consistently favouring smaller states

Degressive Proportionality

1. if $v_i > v_j$ then $w_i = s_i/v_i \leq s_j/v_j = w_j$
   larger states have less voting power
2. if $v_i > v_j$ then $s_i \geq s_j$
   a larger state has at least the seats of a smaller state

Boundary Conditions: $S \leq 751, \; 6 \leq s_i \leq 96$

See definitions and debate in the special issue “Around the Cambridge Compromise” [*Mathematical Social Science* 63(2)]

Degressive Proportionality (2)

We can include DP into the underlying *IP* problem

Assume states sorted in decreasing order of population:
$i < j \iff v_i > v_j$

$$
(IP_{DP}) = \begin{cases} \min \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} & \\ \sum_{i=1}^{n} s_i = S & \\ v_i \cdot s_j - v_j \cdot s_i = d_{ij} & 1 \leq i < j \leq n \\ s_1 = 96 & \\ s_n = 6 & \\ d, s \text{ integer} & \end{cases}
$$

Degressive Proportionality (3)

The model IP_{DP} is quite easily solvable (a few seconds)

The reason is that degressive proportionalty is a *strong requirement*

- any s_i induces a lower bound on s_{i+1}, and so on...

Example: should DP (without boundary conditions) be adopted in the US, California seats would move from 53 to 28

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- any s_i induces a lower bound on s_{i+1}, and so on...

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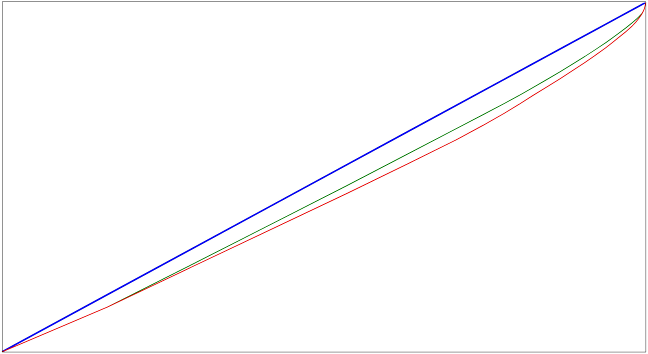
(luckily, Arnold Schwarzenegger is no longer the Governor...)

Degressive Proportionality (4)

We can now evaluate the apportionment found via the currently adopted method, the so-called “Cambridge Compromise”

Plot of Lorentz curves (horizontally stretched)



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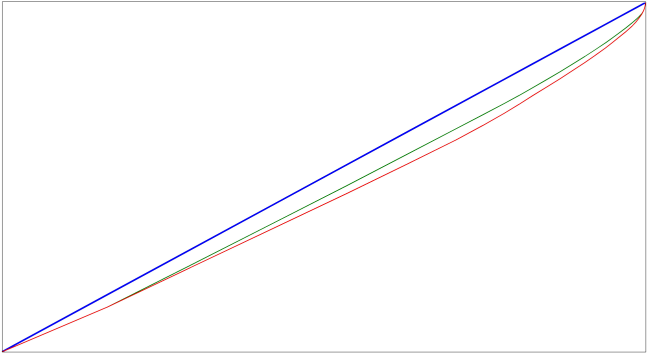

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. . . people at Cambridge did a reasonably good job

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Direction 1 devise a method finding the minimum G apportionment in polynomial time in n and S (and may be $\log V$)

Direction 2 prove that finding the minimum G apportionment is NP-hard

The End

Thanks for your fair attention!